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$\mathrm{SL}(n)$-Covariant L_p-Minkowski Valuations

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Abstract

All continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations defined on convex bodies are completely classified. The L_p-moment body operators turn out to be the nontrivial prototypes of such maps.

1 Introduction

A valuation is a finitely additive function on convex bodies, i.e. nonempty compact convex subsets of $\mathbb{R}^n$. More general, let $\mathcal{Q}^n$ be a subset of $\mathcal{K}^n$, the set of convex bodies in $\mathbb{R}^n$, and let A be an abelian monoid. A map $\Phi\colon \mathcal{Q}^n \to A$ is called a valuation, if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L) \tag{1}$$

for all $K, L \in \mathcal{Q}^n$ with $K \cup L, K \cap L \in \mathcal{Q}^n$. At the beginning of the last century Dehn's solution of Hilbert's Third Problem sparked a lot of interest in valuations. In the 1950's Hadwiger started a systematic study of valuations, which resulted in his famous classification of all continuous rigid motion invariant real valued valuations (see e.g. [40]). Numerous results in the spirit of Hadwiger's theorem were established over the last years (see e.g. [1–4, 22, 25, 29, 30, 36]).

Apart from real valued valuations convex body valued valuations are the focus of increased attention (see e.g. [13–15, 17, 20, 24, 26–28, 31, 38, 41, 43, 44, 47]). The simplest way to define an addition on $\mathcal{K}^n$ such that it becomes an abelian monoid is Minkowski addition, i.e. $K + L := \{x + y : x \in K, y \in L\}$ for $K, L \in \mathcal{K}^n$. The corresponding Minkowski valuations and their generalizations attracted a lot of attention in recent years, because they have applications in many areas such as convexity, stochastic geometry, functional analysis and geometric tomography (see e.g. [5–8, 10, 18, 19, 21, 31, 32, 34, 35, 38, 42, 44, 45, 47–50]).

This article focuses on valuations Φ which are $\mathrm{SL}(n)$-covariant, i.e. $\Phi(\phi K) = \phi\, \Phi\, K$ for all $K \in \mathcal{K}^n$ and $\phi \in \mathrm{SL}(n)$. Clearly the identity $K \mapsto K,\ K \in \mathcal{K}^n$, and the reflection at the origin $K \mapsto -K,\ K \in \mathcal{K}^n$, are $\mathrm{SL}(n)$-covariant Minkowski valuations. Another important example is the moment body operator $\mathrm{M}\colon \mathcal{K}^n \to \mathcal{K}^n_o$ defined by

$$h(\mathrm{M}\, K, u) = \int_K |\langle x, u\rangle|\, dx, \quad u \in \mathbb{R}^n,$$

for all $K \in \mathcal{K}^n$, where $\mathcal{K}^n_o$ denotes the set of convex bodies containing the origin and where $h\,(K,u) := \max_{x\in K}\langle x,u\rangle\,,\ u\in\mathbb{R}^n$, is the support function of the convex body K. For a volume normalized convex body this integral can be interpreted as the expectation of a certain mass distribution depending on K and u. Moment bodies (under a different normalization) are also called centroid bodies and were formally defined by Petty [39], but they actually date back to Dupin and Blaschke. The name "centroid body" comes from the fact that for a symmetric convex body K the boundary of $\mathrm{M}\,K$ consists of points which are, up to normalization, the centroids of K intersected with halfspaces. Ludwig [27] showed that the only nontrivial examples of continuous homogeneous $\mathrm{SL}(n)$-covariant Minkowski valuations on $\mathcal{K}^n_o$ are the moment body operators. Assuming $\mathrm{SL}(n)$-covariance together with homogeneity is basically assuming $\mathrm{GL}(n)$-covariance, where one has to take the determinant of $\phi\in\mathrm{GL}(n)$ into account. Once the valuations that were $\mathrm{GL}(n)$-covariant were understood, the next step was in trying to classify those valuations that were *only* $\mathrm{SL}(n)$-covariant. This turned out to be difficult, because the involved functional equations were a lot more complicated. A real breakthrough here was first achieved by Haberl in [15], where he removed the assumption on homogeneity in Ludwig's classification.

A generalization of Minkowski addition is L_p-Minkowski addition (also known as Minkowski-Firey L_p-addition) for $p>1$, which is defined by

$$h\,(K +_p L, .)^p = h\,(K, .)^p + h\,(L, .)^p \tag{2}$$

for all $K, L \in \mathcal{K}^n_o$, the set of convex bodies containing the origin. The corresponding L_p-Minkowski valuations also received a lot of attention in recent years (see e.g. [5, 7, 9, 12, 16, 19, 27, 32, 33, 35, 37, 48, 49]). In particular, they play an important role in new affine Sobolev inequalities (see e.g. [6, 18, 34, 50]).

The identity and the reflection at the origin, both restricted to $\mathcal{K}^n_o$, are also $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations. Furthermore, the (symmetric) L_p-moment body operator $\mathrm{M}_p\colon \mathcal{K}^n \to \mathcal{K}^n_o$ defined by

$$h\,(\mathrm{M}_p\,K, u)^p = \int\limits_K |\,\langle x,u\rangle\,|^p\ dx, \quad u\in\mathbb{R}^n$$

for all $K\in\mathcal{K}^n$, which encodes the p-th moment of the aforementioned mass distribution, is an $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation. For the definitions of the variants M_p^+ and M_p^- see (4). Ludwig [27] also showed that the only nontrivial examples of continuous homogeneous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations on $\mathcal{K}^n_o$ are the L_p-moment body operators. Our first main theorem improves this classification by removing the homogeneity assumption.

Theorem. *Let $n\geq 3$. An operator $\Phi\colon \mathcal{K}^n_o \to \mathcal{K}^n_o$ is a continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation, if and only if there exist constants $c_1, c_2, c_3, c_4 \geq 0$ such that*

$$\Phi\,K = c_1\,\mathrm{M}_p^+\,K +_p c_2\,\mathrm{M}_p^-\,K +_p c_3 K +_p c_4(-K)$$

for all $K\in\mathcal{K}^n_o$.

Wannerer [47] extended Ludwig's characterization from valuations on $\mathcal{K}_o^n$ to valuations on $\mathcal{K}^n$. In this case four additional operators arise. Our second main theorem characterizes continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations defined on $\mathcal{K}^n$. Again, no assumption on homogeneity is needed. In our case only two additional operators arise. For the definitions of $\hat{\mathrm{M}}_p^+$ and $\hat{\mathrm{M}}_p^-$ see (5). Furthermore, K_o is defined as the convex hull of a convex body K and the origin.

Theorem. *Let $n \geq 3$. An operator $\Phi\colon \mathcal{K}^n \to \mathcal{K}_o^n$ is a continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation, if and only if there exist constants $c_1, c_2, c_3, c_4, c_5, c_6 \geq 0$ such that*

$$\Phi\, K = c_1\, \mathrm{M}_p^+\, K +_p c_2\, \mathrm{M}_p^-\, K +_p c_3 \hat{\mathrm{M}}_p^+ K +_p c_4 \hat{\mathrm{M}}_p^- K +_p c_5 K_o +_p c_6(-K_o)$$

for all $K \in \mathcal{K}^n$.

We remark that a complete classification of all continuous $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuations was also recently obtained by the author in [37].

2 Background Material

As a general reference for the material in this section see [11, 23, 40]. For the dimension of the Euclidean space $\mathbb{R}^n$, we will always assume that $n \geq 1$. The standard basis vectors will be denoted by $e_1, \dots, e_n$ and the origin by o. We will often write $x = (x_1, \dots, x_n)^t$ for $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$ the scalar product, the induced norm and the orthogonal complement will be denoted by $\langle x, y \rangle$, $\|x\|$ and $x^\perp$, respectively. The linear, affine and convex hull are denoted by lin, aff and conv, respectively. Furthermore we will assume $p > 1$ throughout this article.

Associated with a convex body K is its dimension $\dim K$, which is equal to the dimension of $\operatorname{aff} K$. If K is an m-dimensional convex body we denote it's m-dimensional volume by $\mathrm{vol}_m\, K$. Most of the time we will not work with valuations defined on $\mathcal{K}^n$ or $\mathcal{K}_o^n$, but with valuations defined on convex polytopes, i.e convex hulls of finite subsets of $\mathbb{R}^n$. The set of all convex polytopes is denoted by $\mathcal{P}^n$ and the subset of all convex polytopes containing the origin by $\mathcal{P}_o^n$.

Apart from Minkowski addition and L_p-Minkowski addition, there is also a scalar multiplication for convex bodies. It is defined by $sK = \{sx : x \in K\}$ for all $K \in \mathcal{K}^n$ and $s \geq 0$. The sets $\mathcal{K}^n$, $\mathcal{K}_o^n$, $\mathcal{P}^n$ and $\mathcal{P}_o^n$ are all closed under Minkowski addition and scalar multiplication. Furthermore, the sets $\mathcal{K}_o^n$ and $\mathcal{P}_o^n$ are also closed under L_p-Minkowski addition. When we talk about continuity, we mean continuity with respect to the Hausdorff metric. The Hausdorff distance between two convex bodies K, L is defined by

$$\delta(K, L) = \min\{\epsilon > 0 : K + \epsilon B^n \subseteq L,\ L + \epsilon B^n \subseteq K\},$$

where B^n is the Euclidean unit ball in $\mathbb{R}^n$. The sets $\mathcal{K}^n$ and $\mathcal{K}_o^n$ are closed in the Hausdorff topology.

We already mentioned the support function of a convex body in the introduction. A convex body is uniquely defined by it's support function. On the other hand, a function

$h \colon \mathbb{R}^n \to \mathbb{R}$ is the support function of a convex body, if and only if it is sublinear, i.e.

$$h(u+v) \leq h(u) + h(v) \quad \text{and} \quad h(su) = sh(u)$$

for all $u, v \in \mathbb{R}^n$ and $s > 0$. The first property is called subadditivity and the second is 1-homogeneity. To see that the definition of L_p-Minkowski addition (2) makes sense one just has to verify that $h\,(K +_p L, .)$ is a nonnegative sublinear function for all $K, L \in \mathcal{K}_o^n$. Note that every sublinear function is convex and therefore continuous.

The map $K \mapsto h\,(K, .)$ is a homomorphism from $\mathcal{K}^n$ to the space of 1-homogeneous continuous functions on $\mathbb{R}^n$, denoted $C_1(\mathbb{R}^n)$, i.e.

$$h\,(K + L, .) = h\,(K, .) + h\,(L, .) \quad \text{and} \quad h\,(sK, .) = sh\,(K, .)$$

for all $K, L \in \mathcal{K}^n$ and $s \geq 0$. Analogously the map $K \mapsto h\,(K, .)^p$ is a homomorphism from $\mathcal{K}_o^n$ to the space of p-homogeneous continuous functions on $\mathbb{R}^n$, denoted $C_p(\mathbb{R}^n)$, i.e.

$$h\,(K +_p L, .)^p = h\,(K, .)^p + h\,(L, .)^p \quad \text{and} \quad h\,(sK, .)^p = s^p h\,(K, .)$$

for all $K, L \in \mathcal{K}_o^n$ and $s \geq 0$. We can calculate the Hausdorff distance of two convex bodies K, L by $\delta(K, L) = \|h\,(K, .) - h\,(L, .)\,\|_\infty$, where $\|.\|_\infty$ denotes the maximum norm on the Euclidean unit sphere in $\mathbb{R}^n$, denoted S^{n-1}. Since support functions are homogeneous, they are determined by their values on S^{n-1}. Therefore, it makes sense to equip $C_1(\mathbb{R}^n)$ and $C_p(\mathbb{R}^n)$ with this norm. This also makes the two homomorphisms from above continuous.

We have already defined the notion of valuation in the introduction (1). Note that if $\mathcal{Q}^n$ equals $\mathcal{K}^n$,$\mathcal{K}_o^n$, $\mathcal{P}$ or $\mathcal{P}_o^n$ the definition simplifies a little, because in this case $K \cup L \in \mathcal{Q}^n$ implies $K \cap L \in \mathcal{Q}^n$ for all $K, L \in \mathcal{Q}^n$. It is easy to see that a map $\Phi \colon \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski valuation, if and only if the map $K \mapsto h\,(\Phi\, K, .)$ from $\mathcal{K}^n$ to $C_1(\mathbb{R}^n)$ is a valuation. Similarly a map $\Phi \colon \mathcal{K}^n \to \mathcal{K}_o^n$ is an L_p-Minkowski valuation, if and only if the map $K \mapsto h\,(\Phi\, K, .)^p$ from $\mathcal{K}^n$ to $C_p(\mathbb{R}^n)$ is a valuation. The same also holds for valuations defined on $\mathcal{K}_o^n$, $\mathcal{P}$ or $\mathcal{P}_o^n$.

We now recall some general results on valuations. The first theorem is due to Volland [46] (see also [23]). Short proofs of the other results can be found in [37].

It is convenient to assume that A contains an identity element denoted 0 and to define that $\Phi(\emptyset) = 0$, even though $\emptyset \notin \mathcal{K}^n$. We will do this throughout this article.

2.1 Theorem. *Let A be an abelian group and let $\Phi \colon \mathcal{P}^n \to A$ be a valuation. Then Φ satisfies the inclusion exclusion principle, i.e.*

$$\Phi(P_1 \cup \ldots \cup P_m) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,m\}} (-1)^{|S|-1}\, \Phi\left(\bigcap_{i \in S} P_i\right)$$

for all $m \in \mathbb{N}$ and $P_1, \ldots, P_m \in \mathcal{P}^n$ with $P_1 \cup \ldots \cup P_m \in \mathcal{P}^n$.

The convex hull of $k+1$ affinely independet points is called a k-dimensional simplex. Special simplices are the n-dimensional standard simplex $T^n := \operatorname{conv}\{o, e_1, \ldots, e_n\}$ and $\widetilde{T}^{n-1} := \operatorname{conv}\{e_1, \ldots, e_n\}$, which is an $(n-1)$-dimensional simplex.

2.2 Lemma. *Let A be an abelian group and let $\Phi\colon \mathcal{P}^n \to A$ be a valuation. Then Φ is determined by its values on n-dimensional simplices.*

2.3 Lemma. *Let A be an abelian group and let $\Phi\colon \mathcal{P}^n_o \to A$ be a valuation. Then Φ satisfies the inclusion exclusion principle, i.e.*

$$\Phi(P_1 \cup \ldots \cup P_m) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,m\}} (-1)^{|S|-1}\, \Phi\left(\bigcap_{i\in S} P_i\right)$$

for all $m \in \mathbb{N}$ and $P_1, \ldots, P_m \in \mathcal{P}^n_o$ with $P_1 \cup \ldots \cup P_m \in \mathcal{P}^n_o$.

2.4 Lemma. *Let A be an abelian group and let $\Phi\colon \mathcal{P}^n_o \to A$ be a valuation. Then Φ is determined by its values on n-dimensional simplices with one vertex at the origin and its value on $\{o\}$.*

A valuation is called simple, if $\Phi\, K = 0$ for all $K \in \mathcal{Q}^n$ with $\dim K < n$.

2.5 Lemma. *Let A be an abelian group and let $\Phi\colon \mathcal{P}^n \to A$ be a simple valuation. Then Φ is determined by its values on $\mathcal{P}^n_o$.*

Finally we recall Cauchy's functional equation

$$f(a+b) = f(a) + f(b) \quad \forall a, b \in \mathbb{R}. \tag{3}$$

Of course every linear function satisfies (3). It is a well known fact that a nonlinear function $f\colon \mathbb{R} \to \mathbb{R}$ can only satisfy (3), if it has a dense graph in $\mathbb{R}^2$ or equivalently if every open subset of $\mathbb{R}$ has a dense image under f. A function $f\colon (0, +\infty) \to \mathbb{R}$ satisfying (3) for all $a, b \in (0, +\infty)$ can be extendend to an odd function on $\mathbb{R}$, which then satisfies Cauchy's functional equation for all $a, b \in \mathbb{R}$. Therefore such a function f is either linear or it has the property that every open subset of $(0, +\infty)$ has a dense image under f.

3 $\mathrm{SL}(n)$-Covariance

Let $\mathcal{Q}^n$ be $\mathcal{K}^n$, $\mathcal{P}^n$, $\mathcal{K}^n_o$ or $\mathcal{P}^n_o$. A map $\Phi\colon \mathcal{Q}^n \to \mathcal{K}^n_o$ is called $\mathrm{SL}(n)$-covariant, if it satisfies

$$\Phi(\phi K) = \phi\, \Phi\, K$$

for all $K \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}(n)$. A map $\Phi\colon \mathcal{Q}^n \to C_p(\mathbb{R}^n)$ is called $\mathrm{SL}(n)$-covariant, if it satisfies

$$\Phi(\phi K) = \Phi(K) \circ \phi^t$$

for all $K \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}(n)$. Since

$$h\left(\phi\, \Phi\, K, u\right) = h\left(\Phi\, K, \phi^t u\right)$$

holds for all $K \in \mathcal{Q}^n$, $u \in \mathbb{R}^n$ and $\phi \in \mathrm{SL}(n)$, we see that a map $\Phi\colon \mathcal{Q}^n \to \mathcal{K}^n_o$ is $\mathrm{SL}(n)$-covariant, if and only if $K \mapsto h\left(\Phi\, K, .\right)^p$ from $\mathcal{Q}^n$ to $C_p(\mathbb{R}^n)$ is $\mathrm{SL}(n)$-covariant. We also make analogous definitions and remarks for the general linear group, $\mathrm{GL}(n)$.

We define an $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation $\mathrm{I}_p^+ \colon \mathcal{K}^n \to \mathcal{K}_o^n$ by

$$h\left(\mathrm{I}_p^+ K, .\right)^p = \max_{x \in K} \langle x, . \rangle_+^p$$

for all $K \in \mathcal{K}^n$, where $\langle x, . \rangle_+ := \max(0, \langle x, . \rangle)$ denotes the positive part of $\langle x, . \rangle$. Note that $\mathrm{I}_p^+ K = K_o$ for all $K \in \mathcal{K}^n$. Similarly we define an $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation $\mathrm{I}_p^- \colon \mathcal{K}^n \to \mathcal{K}_o^n$ by

$$h\left(\mathrm{I}_p^- K, .\right)^p = \max_{x \in K} \langle x, . \rangle_-^p$$

for all $K \in \mathcal{K}^n$, where $\langle x, . \rangle_- := \max(0, -\langle x, . \rangle)$ denotes the negative part of $\langle x, . \rangle$. Note that $\mathrm{I}_p^- K = -K_o$ for all $K \in \mathcal{K}^n$. There are two other very similar $\mathrm{SL}(n)$-covariant valuations, namely $\mathrm{J}_p^+ \colon \mathcal{K}^n \to C_p(\mathbb{R}^n)$ and $\mathrm{J}_p^- \colon \mathcal{K}^n \to C_p(\mathbb{R}^n)$. These are defined by

$$\mathrm{J}_p^+ K = \min_{x \in K} \langle x, . \rangle_+^p$$

for all $K \in \mathcal{K}^n$ and by

$$\mathrm{J}_p^- K = \min_{x \in K} \langle x, . \rangle_-^p$$

for all $K \in \mathcal{K}^n$, respectively. Note that $\mathrm{J}_p^+ K$ and $\mathrm{J}_p^- K$ are not necessarily support functions of convex bodies. Also note that J_p^+ and J_p^- vanish on $\mathcal{K}_o^n$. Finally we remark that I_p^+, I_p^-, J_p^+ and J_p^- are also continuous and $\mathrm{GL}(n)$-covariant.

Another family of $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations are the L_p-moment body operators. We define $\mathrm{M}_p^+ \colon \mathcal{K}^n \to \mathcal{K}_o^n$ by

$$h\left(\mathrm{M}_p^+ K, .\right)^p = \int_K \langle x, . \rangle_+^p \, dx \tag{4}$$

for all $K \in \mathcal{K}^n$. Similarly we define $\mathrm{M}_p^- \colon \mathcal{K}^n \to \mathcal{K}_o^n$. Note that M_p^+ and M_p^- are simple. A variant of M_p^+ is $\hat{\mathrm{M}}_p^+ \colon \mathcal{K}^n \to \mathcal{K}_o^n$ defined by

$$h\left(\hat{\mathrm{M}}_p^+ K, .\right)^p = \int_{K_o \setminus K} \langle x, . \rangle_+^p \, dx \tag{5}$$

for all $K \in \mathcal{K}^n$. Similarly we define $\hat{\mathrm{M}}_p^- \colon \mathcal{K}^n \to \mathcal{K}_o^n$. Note that $\hat{\mathrm{M}}_p^+$ and $\hat{\mathrm{M}}_p^-$ vanish on $\mathcal{K}_o^n$. Finally we remark that M_p^+, M_p^-, $\hat{\mathrm{M}}_p^+$ and $\hat{\mathrm{M}}_p^-$ are also continuous and $(n+p)/p$-homogeneous.

If we view $\mathcal{K}_o^n$ as a subset of $C_p(\mathbb{R}^n)$, then we have defined eight linearly independent $\mathrm{SL}(n)$-covariant valuations with values in $C_p(\mathbb{R}^n)$. Notice that evaluating the minus-version of one of these operators at K is the same as evaluating the plus-version at $-K$, for example $\mathrm{I}_p^- K = \mathrm{I}_p^+(-K)$.

For later use we need to calculate some constants. Let $i \in \{1, \dots, n\}$. We start with

$$h\left(\mathrm{I}_p^+[o, e_i], x\right)^p = \max_{y \in [o, e_i]} \langle y, x \rangle_+^p = \langle e_i, x \rangle_+^p = (x_i)_+^p \tag{6}$$

for all $x \in \mathbb{R}^n$.

Next we calculate

$$
\begin{aligned}
h\left(\mathrm{M}_p^+ T^n, e_i\right)^p &= \int\limits_{T^n} \langle x, e_i\rangle_+^p \, dx \\
&= \int\limits_0^1 x_i^p \operatorname{vol}_{n-1}\left((1-x_i)T^{n-1}\right) dx_i \\
&= \operatorname{vol}_{n-1}\left(T^{n-1}\right) \int\limits_0^1 x_i^p (1-x_i)^{n-1} dx_i \\
&= \frac{1}{(n-1)!} B(p+1, n) \\
&= \frac{\Gamma(p+1)\Gamma(n)}{(n-1)!\ \Gamma(p+1+n)} \\
&= \frac{\Gamma(p+1)}{\Gamma(p+1+n)},
\end{aligned} \tag{7}
$$

where B and Γ denote the Beta function and the Gamma function, respectively. Obviously we have

$$h\left(\mathrm{M}_p^+ T^n, -e_i\right)^p = 0. \tag{8}$$

Now we will calculate some constants which will be used in the classification of valuations on $\mathcal{P}^n$. Let $i \in \{1, \dots, n\}$. We start with

$$h\left(\mathrm{I}_p^+ [e_i, 2e_i], x\right)^p = \max_{y \in [e_i, 2e_i]} \langle y, x\rangle_+^p = \langle 2e_i, x\rangle_+^p = 2^p (x_i)_+^p \tag{9}$$

for all $x \in \mathbb{R}^n$. Analogously we see that

$$\mathrm{J}_p^+([e_i, 2e_i])(x) = (x_i)_+^p. \tag{10}$$

Next we calculate

$$h\left(\hat{\mathrm{M}}_p^+ \widetilde{T}^{n-1}, e_i\right)^p = h\left(\mathrm{M}_p^+ T^n, e_i\right)^p = \frac{\Gamma(p+1)}{\Gamma(p+1+n)} \tag{11}$$

and

$$h\left(\hat{\mathrm{M}}_p^+ \widetilde{T}^{n-1}, -e_i\right)^p = 0. \tag{12}$$

Now let $n \geq 2$. We set $K = \operatorname{conv}\{e_1, e_2, e_1 + e_2\}$ and calculate:

$$
\begin{gathered}
h\left(\mathrm{I}_p^+ K, e_1\right)^p = 1, \quad h\left(\mathrm{I}_p^+ K, e_2\right)^p = 1, \quad h\left(\mathrm{I}_p^+ K, e_1 + e_2\right)^p = 2^p, \\
\mathrm{J}_p^+(K)(e_1) = 0, \quad \mathrm{J}_p^+(K)(e_2) = 0, \quad \mathrm{J}_p^+(K)(e_1 + e_2) = 1.
\end{gathered} \tag{13}
$$

The values of $h\left(\mathrm{I}_p^- K, .\right)^p$ and $\mathrm{J}_p^- K$ in these directions are all equal to 0. Finally we compute:

$$\begin{aligned} h\left(\mathrm{I}_p^+ \widetilde{T}^1, e_1\right)^p = 1, \quad h\left(\mathrm{I}_p^+ \widetilde{T}^1, e_1+e_2\right)^p = 1, \quad h\left(\mathrm{I}_p^+ \widetilde{T}^1, 2e_1+e_2\right)^p = 2^p, \\ \mathrm{J}_p^+(\widetilde{T}^1)(e_1) = 0, \quad \mathrm{J}_p^+(\widetilde{T}^1)(e_1+e_2) = 1, \quad \mathrm{J}_p^+(\widetilde{T}^1)(2e_1+e_2) = 1. \end{aligned} \tag{14}$$

The values of $h\left(\mathrm{I}_p^- \widetilde{T}^1, .\right)^p$ and $\mathrm{J}_p^- \widetilde{T}^1$ in these directions are all equal to 0.

In $\mathbb{R}^2$ there are other $\mathrm{SL}(n)$-covariant valuations, which do not show up in $\mathbb{R}^n$ for $n \geq 3$. We will only cover those additional operators in $\mathbb{R}^2$ that we need for the proof of the $n \geq 3$ case. We define $\mathrm{E}_p^+ \colon \mathcal{K}^2 \to \mathcal{K}_o^2$ by

$$h\left(\mathrm{E}_p^+ K, .\right)^p = \frac{1}{2} \sum_{\substack{u \in S^1 \\ h(K,u)=0}} \max_{x \in F(K,u)} \langle x, . \rangle_+^p$$

for all $K \in \mathcal{K}^n$, where $F(K,u) := \{x \in K : \langle x, u \rangle = h(K,u)\}$. Note that this sum has at most two nonzero summands. Clearly E_p^+ is $\mathrm{GL}(n)$-covariant. The valuation property can be shown by a case-by-case analysis. Similarly we define $\mathrm{E}_p^- \colon \mathcal{K}^2 \to \mathcal{K}_o^2$. It is easy to see that E_p^+ and E_p^- coincide with I_p^+ and I_p^-, respectively, for 1-dimensional convex bodies K with $o \in \operatorname{aff} K$.

Finally $\mathrm{F}_p^+ \colon \mathcal{K}^2 \to C_p(\mathbb{R}^2)$ is defined by

$$\mathrm{F}_p^+ K = \frac{1}{2} \sum_{\substack{u \in S^1 \\ h(K,u)=0}} \min_{x \in F(K,u)} \langle x, . \rangle_+^p$$

for all $K \in \mathcal{K}^n$. Similarly we define $\mathrm{F}_p^- \colon \mathcal{K}^2 \to C_p(\mathbb{R}^2)$. Note that $\mathrm{F}_p^+ K$ and $\mathrm{F}_p^- K$ are not necessarily support functions of convex bodies. Also note that F_p^+ and F_p^- vanish on $\mathcal{K}_o^n$. It is easy to see that F_p^+ and F_p^- coincide with J_p^+ and J_p^-, respectively, for 1-dimensional convex bodies K with $o \in \operatorname{aff} K$. Notice again how one can get the minus-version of an operator by inserting $-K$ in the plus-version.

We will now collect some properties of these new valuations in $\mathbb{R}^2$ and calculate some constants. Let $i \in \{1,2\}$. We start with calculating

$$h\left(\mathrm{E}_p^+ T^2, e_i\right)^p = \frac{1}{2}\left(h\left([0,e_1],e_i\right)^p + h\left([0,e_2],e_i\right)^p\right) = \frac{1}{2}.$$

Similarly we have

$$h\left(\mathrm{E}_p^+ T^2, -e_i\right)^p = 0.$$

Therefore we get:

$$\begin{aligned} h\left(\mathrm{I}_p^+ T^2, e_i\right)^p - h\left(\mathrm{E}_p^+ T^2, e_i\right)^p = 1 - \frac{1}{2} = \frac{1}{2}, \\ h\left(\mathrm{I}_p^+ T^2, -e_i\right)^p - h\left(\mathrm{E}_p^+ T^2, -e_i\right)^p = 0 - 0 = 0. \end{aligned} \tag{15}$$

Next, we will look at the continuity of the above operators. Clearly

$$\lim_{\varepsilon\to 0}[-\varepsilon e_1, e_1] + [-\varepsilon e_2, \varepsilon e_2] = [0, e_1].$$

Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0} h\left(\mathrm{I}_p^+\left([-\varepsilon e_1, e_1] + [-\varepsilon e_2, \varepsilon e_2]\right), e_1\right)^p - h\left(\mathrm{E}_p^+\left([-\varepsilon e_1, e_1] + [-\varepsilon e_2, \varepsilon e_2]\right), e_1\right)^p \\
=&\lim_{\varepsilon\to 0} h\left([-\varepsilon e_1, e_1] + [-\varepsilon e_2, \varepsilon e_2], e_1\right)^p \\
=&h\left([o, e_1], e_1\right)^p \\
=&1
\end{aligned}$$

and since

$$h\left(\mathrm{I}_p^+[0, e_1], e_1\right)^p - h\left(\mathrm{E}_p^+[0, e_1], e_1\right)^p = 0,$$

we see that

$$P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p$$

from $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$ is not continuous at $[o, e_1]$. Similarly we prove the following lemma.

3.1 Lemma. *The only linear combination of*

$$P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p \quad \textit{and} \quad P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p$$

which is continuous at $[o, e_1]$ *is the trivial one.*

Finally we need to calculate some constants related to valuations on $\mathcal{P}^2$. Let $i \in \{1, 2\}$. We start with

$$h\left(\mathrm{E}_p^+ \widetilde{T}^1, e_i\right)^p = \frac{1}{2}\left(\langle e_1, e_i\rangle_+^p + \langle e_2, e_i\rangle_+^p\right) = \frac{1}{2}.$$

Similarly we have

$$h\left(\mathrm{E}_p^+ \widetilde{T}^1, -e_i\right)^p = 0.$$

Therefore we get:

$$\begin{aligned}
h\left(\mathrm{I}_p^+ \widetilde{T}^1, e_i\right)^p - h\left(\mathrm{E}_p^+ \widetilde{T}^1, e_i\right)^p &= 1 - \frac{1}{2} = \frac{1}{2}, \\
h\left(\mathrm{I}_p^+ \widetilde{T}^1, -e_i\right)^p - h\left(\mathrm{E}_p^+ \widetilde{T}^1, -e_i\right)^p &= 0 - 0 = 0.
\end{aligned} \tag{16}$$

Analogously we calculate:

$$\begin{aligned}
h\left(\mathrm{I}_p^+ T^2, e_i\right)^p - h\left(\mathrm{E}_p^+ T^2, e_i\right)^p + \mathrm{J}_p^+(T^2)(e_i) - \mathrm{F}_p^+(T^2)(e_i) &= \frac{1}{2}, \\
h\left(\mathrm{I}_p^+ T^2, -e_i\right)^p - h\left(\mathrm{E}_p^+ T^2, -e_i\right)^p + \mathrm{J}_p^+(T^2)(-e_i) - \mathrm{F}_p^+(T^2)(-e_i) &= 0.
\end{aligned} \tag{17}$$

Similar to the proof of Lemma 3.1 we see that the only linear combination of the four operators

$$\begin{aligned}
&P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p,\\
&P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p,\\
&P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p + \mathrm{J}_p^+ P - \mathrm{F}_p^+ P
\end{aligned}$$

and

$$P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p + \mathrm{J}_p^- P - \mathrm{F}_p^- P$$

which is continuous at $[o, e_1]$ is actually a linear combination of $\mathrm{J}_p^+ - \mathrm{F}_p^+$ and $\mathrm{J}_p^- - \mathrm{F}_p^-$. Clearly

$$\lim_{\varepsilon \to 0} \varepsilon e_2 + [o, e_1] = [0, e_1].$$

Since

$$\lim_{\varepsilon \to 0} (\mathrm{J}_p^+ - \mathrm{F}_p^+)(\varepsilon e_2 + [o, e_1])(e_1) = \lim_{\varepsilon \to 0} 0 - \frac{1}{2} = -\frac{1}{2}$$

and since

$$(\mathrm{J}_p^+ - \mathrm{F}_p^+)([0, e_1])(e_1) = 0,$$

we see that $\mathrm{J}_p^+ - \mathrm{F}_p^+$ is not continuous at $[o, e_1]$. Similarly we see that the only linear combination of the operators $\mathrm{J}_p^+ - \mathrm{F}_p^+$ and $\mathrm{J}_p^- - \mathrm{F}_p^-$ which is continuous at $[0, e_1]$ is the trivial one. Therefore we arrive at the following lemma.

3.2 Lemma. *The only linear combination of*

$$\begin{aligned}
&P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p,\\
&P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p,\\
&P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p + \mathrm{J}_p^+ P - \mathrm{F}_p^+ P
\end{aligned}$$

and

$$P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p + \mathrm{J}_p^- P - \mathrm{F}_p^- P$$

which is continuous at $[o, e_1]$ *is the trivial one.*

We complete this section with a simple lemma (cf. [37]).

3.3 Lemma. *Let $\mathcal{Q}^n$ be either $\mathcal{P}_o^n$ or $\mathcal{P}^n$ and let $\Phi\colon \mathcal{Q}^n \to C_p(\mathbb{R}^n)$ be $\mathrm{SL}(n)$-covariant. Furthermore let $\phi \in \mathrm{GL}(n)$ with $\det \phi > 0$. Then*

$$\Phi(\phi P) = \det(\phi)^{-\frac{p}{n}} \, \Phi\left(\det(\phi)^{\frac{1}{n}} P\right) \circ \phi^t$$

for all $P \in \mathcal{Q}^n$.

Proof. Since $\det(\phi)^{-\frac{1}{n}} \phi \in \mathrm{SL}(n)$, this follows directly from the $\mathrm{SL}(n)$-covariance of Φ and the p-homogeneity of the functions in $C_p(\mathbb{R}^n)$. □

4 Main Results on $\mathcal{K}_o^n$

The goal of this section is the classification of all continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations on $\mathcal{K}_o^n$. It will be convenient to first prove a slightly more general theorem about valuations from $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$. We begin by proving a classification with the additional assumption of simplicity. The next theorem is an adaptation of a corresponding theorem concerning $\mathrm{SL}(n)$-contravariant valuations from [37].

4.1 Theorem. *Let $n \geq 3$ and let $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ be a simple $\mathrm{SL}(n)$-covariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0,+\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$. Then there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\Phi\, P = c_1 h\left(\mathrm{M}_p^+\, P, .\right)^p + c_2 h\left(\mathrm{M}_p^-\, P, .\right)^p \tag{18}$$

for all $P \in \mathcal{P}_o^n$.

Proof. Using Lemma 3.3 and Lemma 2.4 it is enough to prove

$$\Phi(sT^n) = c_1 h\left(\mathrm{M}_p^+(sT^n), .\right)^p + c_2 h\left(\mathrm{M}_p^-(sT^n), .\right)^p \tag{19}$$

for $s > 0$.

1. Functional Equation: Let $\lambda \in (0,1)$ and denote by H_λ the hyperplane through o with normal vector $\lambda e_1 - (1-\lambda)e_2$. Since Φ is a valuation we get

$$\Phi(sT^n) + \Phi(sT^n \cap H_\lambda) = \Phi(sT^n \cap H_\lambda^+) + \Phi(sT^n \cap H_\lambda^-),$$

where H_λ^+ and H_λ^- are the two halfspaces bounded by H_λ. Because Φ is assumed to be simple, we get

$$\Phi(sT^n) = \Phi(sT^n \cap H_\lambda^+) + \Phi(sT^n \cap H_\lambda^-). \tag{20}$$

Define $\phi_\lambda \in \mathrm{GL}(n)$ by

$$\phi_\lambda e_1 = e_1, \quad \phi_\lambda e_2 = (1-\lambda)e_1 + \lambda e_2, \quad \phi_\lambda e_k = e_k \quad \text{for } 3 \leq k \leq n$$

and $\psi_\lambda \in \mathrm{GL}(n)$ by

$$\psi_\lambda e_1 = (1-\lambda)e_1 + \lambda e_2, \quad \psi_\lambda e_2 = e_2, \quad \psi_\lambda e_k = e_k \quad \text{for } 3 \leq k \leq n.$$

Note that

$$\det(\phi_\lambda) = \lambda \quad \text{and} \quad \det(\psi_\lambda) = 1-\lambda. \tag{21}$$

Since

$$T^n \cap H_\lambda^+ = \phi_\lambda T^n \quad \text{and} \quad T^n \cap H_\lambda^- = \psi_\lambda T^n,$$

Equation (20) becomes

$$\Phi(sT^n) = \Phi(\phi_\lambda sT^n) + \Phi(\psi_\lambda sT^n).$$

Using Lemma 3.3 and (21) we can rewrite the last equation as

$$\Phi(sT^n)(x) = \lambda^{-\frac{p}{n}}\,\Phi\left(\lambda^{\frac{1}{n}}sT^n\right)(\phi_\lambda^t x) + (1-\lambda)^{-\frac{p}{n}}\,\Phi\left((1-\lambda)^{\frac{1}{n}}sT^n\right)(\psi_\lambda^t x) \tag{22}$$

for all $x \in \mathbb{R}^n$.

2. Homogeneity: For $y \in \{e_1, e_2\}^\perp$ (22) becomes

$$\Phi(sT^n)(y) = \lambda^{-\frac{p}{n}}\,\Phi\left(\lambda^{\frac{1}{n}}sT^n\right)(y) + (1-\lambda)^{-\frac{p}{n}}\,\Phi\left((1-\lambda)^{\frac{1}{n}}sT^n\right)(y).$$

Replace s with $s^{\frac{1}{n}}$ in the above equation and define $g(s) = \Phi\left(s^{\frac{1}{n}}T^n\right)(y)$. Then we have

$$g(s) = \lambda^{-\frac{p}{n}}g(\lambda s) + (1-\lambda)^{-\frac{p}{n}}g((1-\lambda)s).$$

Let $a, b > 0$. We set $s = a+b$ and $\lambda = \frac{a}{a+b}$ to get

$$g(a+b) = \left(\frac{a}{a+b}\right)^{-\frac{p}{n}} g(a) + \left(\frac{b}{a+b}\right)^{-\frac{p}{n}} g(b)$$

and hence

$$(a+b)^{-\frac{p}{n}}g(a+b) = a^{-\frac{p}{n}}g(a) + b^{-\frac{p}{n}}g(b).$$

We see that $s \mapsto s^{-\frac{p}{n}}g(s)$ solves Cauchy's functional equation for $s > 0$. By assumption there is a bounded open interval I_y such that $g(I_y)$ is not dense in $\mathbb{R}$. It follows that $s \mapsto s^{-\frac{p}{n}}g(s)$ is linear. This implies $s^{-\frac{p}{n}}g(s) = sg(1)$ and hence $g(s) = s^{1+\frac{p}{n}}g(1)$. The definition of g yields

$$\Phi(sT^n)(y) = g(s^n) = s^{n+p}g(1) = s^{n+p}\,\Phi(T^n)(y).$$

Since $n \geq 3$ and since we can do the above calculation for any two standard basis vectors, we obtain in particular

$$\Phi(sT^n)(\pm e_i) = s^{n+p}\,\Phi(T^n)(\pm e_i) \quad \text{for } i = 1, \dots, n. \tag{23}$$

3. Constants: Let $i \in \{1, \dots, n\}$. Since $n \geq 3$, we can find a permutation of the coordinates $\phi \in \mathrm{SL}(n)$ such that $\phi^t e_1 = e_i$. It follows that

$$\Phi(T^n)(e_i) = \Phi(T^n)(\phi^t e_1) = \Phi(\phi T^n)(e_1) = \Phi(T^n)(e_1). \tag{24}$$

Similarly we get $\Phi(T^n)(-e_i) = \Phi(T^n)(-e_1)$. Set

$$c_1 = \frac{\Gamma(p+1+n)}{\Gamma(p+1)}\,\Phi(T^n)(e_1) \quad \text{and} \quad c_2 = \frac{\Gamma(p+1+n)}{\Gamma(p+1)}\,\Phi(T^n)(-e_1). \tag{25}$$

4. Induction: We are now going to show by induction on the number m of coordinates of x not equal to zero that

$$\Phi(sT^n)(x) = c_1 h\left(\mathrm{M}_p^+(sT^n), x\right)^p + c_2 h\left(\mathrm{M}_p^-(sT^n), x\right)^p \tag{26}$$

for $s>0$ and for all $x\in\mathbb{R}^n$. Note that since $P\mapsto c_1h\left(\mathrm{M}_p^+(P),.\right)^p+c_2h\left(\mathrm{M}_p^-(P),.\right)^p$ satisfies the assumptions of the theorem it also satisfies (22) and (23).

The case $m=0$ is trivial. The case $m=1$ is also easy to verify with (23), (24), (25), (7) and (8). Now, let $m\geq 2$. Without loss of generality assume that $x_1,x_2\neq 0$ and $|x_1|\geq|x_2|$. Since functions in $C_p(\mathbb{R}^n)$ are continuous, we can further assume that $|x_1|>|x_2|$.

First consider the case that x_1 and x_2 have different signs. Set $\lambda=\frac{x_1}{x_1-x_2}\in(0,1)$ and calculate

$$\begin{aligned}\phi_\lambda^t x&=x_1e_1+x_1(1-\lambda)e_2+x_2\lambda e_2+x_3e_3+\ldots+x_ne_n\\&=x_1e_1+x_3e_3+\ldots+x_ne_n.\end{aligned}$$

Similarly we have

$$\psi_\lambda^t x=\left(x_2e_2+x_3e_3+\ldots+x_ne_n\right).$$

Using (22) and the induction hypotheses gives the desired result.

Now consider the case that x_1,x_2 have the same sign. Set $\lambda=1-\frac{x_2}{x_1}\in(0,1)$ and calculate

$$\begin{aligned}&\phi_\lambda^t(x_1e_1+x_3e_3+\ldots+x_ne_n)\\&=x_1e_1+x_1(1-\lambda)e_2+x_3e_3+\ldots+x_ne_n\\&=x_1e_1+x_2e_2+x_3e_3+\ldots+x_ne_n\\&=x\end{aligned}$$

or equivalently

$$\phi_\lambda^{-t}x=x_1e_1+x_3e_3+\ldots+x_ne_n.$$

Similarly we calculate

$$\psi_\lambda^t\phi_\lambda^{-t}x=x_1(1-\lambda)e_1+x_3e_3+\ldots+x_ne_n.$$

Using (22) with x replaced by $\phi_\lambda^{-t}x$ and using the induction hypotheses gives the desired result.

This completes the induction and proves (26) or equivalently (19). □

We now use Theorem 4.1 to rule out the existence of certain valuations. This will be needed in the induction step in the proof of Theorem 4.4.

4.2 Lemma. *Let $n\geq 2$. If $\Phi\colon\mathcal{P}_o^n\to C_p(\mathbb{R}^n)$ is a simple* $\mathrm{GL}(n)$*-covariant valuation which is continuous at the line segment $[0,e_1]$, then $\Phi=0$.*

Proof. First note that the $\mathrm{GL}(n)$-covariance implies p-homogeneity. For $n\geq 3$ the assertion follows directly from Theorem 4.1, since $P\mapsto h\left(\mathrm{M}_p^+P,.\right)^p$ and $P\mapsto h\left(\mathrm{M}_p^-P,.\right)^p$ are both $(n+p)$-homogeneous.

Only the case $n=2$ remains. Using the $\mathrm{GL}(n)$-covariance and Lemma 2.4 we see that Φ is determined by its values on T^2. The $\mathrm{GL}(n)$-covariance and the simplicity of Φ imply

$$\Phi(T^2)(x) = \Phi(T^2)(\phi_\lambda^t x) + \Phi(T^2)(\psi_\lambda^t x). \tag{27}$$

Similar to the steps 3 and 4 in the proof of Theorem 4.1, we see that Φ is determined by the two values $\Phi(T^2)(\pm e_1)$.

Now notice that E_p^+ takes the same values as I_p^+ for all $P \in \mathcal{P}_o^2$ with $\dim P \leq 1$. Therefore

$$P \mapsto h\left(\mathrm{I}_p^+ P, .\right)^p - h\left(\mathrm{E}_p^+ P, .\right)^p$$

is a simple $\mathrm{GL}(n)$-covariant valuation. The same holds for

$$P \mapsto h\left(\mathrm{I}_p^- P, .\right)^p - h\left(\mathrm{E}_p^- P, .\right)^p.$$

Using (15) we see that Φ is a linear combination of these two operators. Since the only operator in the linear hull of these operators which is continuous at the line segment $[0, e_1]$ is $\Phi = 0$ (see Lemma 3.1), the assertion follows. $\square$

The next lemma will also be needed in the proof of Theorem 4.4. It implies that an $\mathrm{SL}(n)$-covariant operator $\Phi\colon \mathcal{P}_o^n \to \mathcal{K}_o^n$ maps a convex polytope which is contained in some linear subspace to a convex body which is contained in the same linear subspace.

4.3 Lemma. *Let $n \geq 2$. If $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ is $\mathrm{SL}(n)$-covariant, then*

$$\Phi(P)(x) = \Phi(P)(\pi_P x)$$

for all $P \in \mathcal{P}_o^n$ and $x \in \mathbb{R}^n$, where π_P denotes the orthogonal projection onto $\operatorname{lin} P$.

Proof. For $\dim P = n$ there is nothing to show. For $\dim P = 0$, i.e. $P = \{o\}$, we have

$$\Phi\, P = \Phi(\phi P) = \Phi(P) \circ \phi^t$$

for all $\phi \in \mathrm{SL}(n)$. Since $n \geq 2$, $\Phi\, P$ must be constant.

Now let $d = \dim P$ with $0 < d < n$. Using the $\mathrm{SL}(n)$-covariance we can assume without loss of generality that $P \subseteq \{e_1, \dots, e_d\}$. Define $\phi \in \mathrm{SL}(n)$ by

$$\phi = \begin{pmatrix} I' & A \\ 0 & I'' \end{pmatrix},$$

where $A \in \mathbb{R}^{d\times(n-d)}$ is an arbitrary matrix, $0 \in \mathbb{R}^{(n-d)\times d}$ is the zero matrix and where $I' \in \mathbb{R}^{d\times d}$ and $I'' \in \mathbb{R}^{(n-d)\times(n-d)}$ are identity matrices. Since $\phi P = P$ and since Φ is $\mathrm{SL}(n)$-covariant, we get

$$\Phi\, P = \Phi(\phi P) = \Phi(P) \circ \phi^t. \tag{28}$$

Write $x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^{n-d}$. Since $\Phi\, P$ is continuous, we can assume that x' is not zero. Note that

$$\phi^t x = \begin{pmatrix} I' & 0^t \\ A^t & I'' \end{pmatrix} \cdot \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} x' \\ A^t x' + x'' \end{pmatrix}. \tag{29}$$

Because we can choose A such that $A^t x' + x''$ is zero, the assertion follows from (28) and (29). $\square$

Now we are able to classify all $\mathrm{SL}(n)$-covariant valuations from $\mathcal{P}_o^n$ to $C_p(\mathbb{R}^n)$ which satisfy certain continuity properties.

4.4 Theorem. *Let $n \geq 3$ and let $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ be an $\mathrm{SL}(n)$-covariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0,+\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$. Also assume that Φ is continuous at the line segment $[0,e_1]$. Then there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that*

$$\Phi\, P = c_1 h\left(\mathrm{M}_p^+\, P, .\right)^p + c_2 h\left(\mathrm{M}_p^-\, P, .\right)^p + c_3 h\left(\mathrm{I}_p^+\, P, .\right)^p + c_4 h\left(\mathrm{I}_p^-\, P, .\right)^p \tag{30}$$

for all $P \in \mathcal{P}_o^n$.

Proof. Lemma 4.3 and the p-homogeneity of functions in $C_p(\mathbb{R}^n)$ show that

$$\Phi([o,e_1])(x) = \Phi([o,e_1])(x_1 e_1) = |x_1|^p\, \Phi([o,e_1])(\mathrm{sgn}(x_1) e_1) \tag{31}$$

for all $x \in \mathbb{R}^n$. Set $c_3 = \Phi([o,e_1])(e_1)$ and $c_4 = \Phi([o,e_1])(-e_1)$. Define $\Psi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ by

$$\Psi\, P = \Phi\, P - c_3 h\left(\mathrm{I}_p^+\, P, .\right)^p - c_4 h\left(\mathrm{I}_p^-\, P, .\right)^p \tag{32}$$

for all $P \in \mathcal{P}_o^n$. Note that Ψ is also an $\mathrm{SL}(n)$-covariant valuation. If we can show that Ψ is simple, then the assertion follows from Theorem 4.1.

We need to prove that $\Psi\, P = 0$ for all $P \in \mathcal{P}_o^n$ with $\dim P \leq n-1$. Using the $\mathrm{SL}(n)$-covariance we can assume without loss of generality that $P \subseteq \mathrm{lin}\{e_1, \ldots, e_{n-1}\}$. We will use induction on $d = 1, \ldots, n-1$ to show that $\Psi\, P = 0$ for all convex polytopes $P \subseteq \mathrm{lin}\{e_1, \ldots, e_d\}$. Define $\hat{\Psi}\colon \mathcal{P}_o^d \to C_p(\mathbb{R}^d)$ by

$$\hat{\Psi} P = \Psi(\iota_d P) \circ \iota_d \tag{33}$$

for all $P \in \mathcal{P}_o^d$, where ι_d denotes the natural embedding of $\mathbb{R}^d$ in $\mathbb{R}^n$. It is easy to see that $\hat{\Psi}$ is a $\mathrm{GL}(d)$-covariant valuation. Since Ψ is $\mathrm{SL}(n)$-covariant, we have $\Psi(\{o\}) = 0$ by Lemma 4.3. For $d = 1$ the induction statement follows from (31), (6) and the $\mathrm{GL}(d)$-covariance. For $2 \leq d \leq n-1$ the induction statement follows from the induction hypothesis, Lemma 4.2 and Lemma 4.3. This finishes the induction and completes the proof of the theorem. □

Finally we can prove our desired result about continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations on $\mathcal{K}_o^n$. This is one direction of the first main theorem from the introduction. The other direction is trivial.

4.5 Corollary. *Let $n \geq 3$. If $\Phi\colon \mathcal{K}_o^n \to \mathcal{K}_o^n$ is a continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation, then there exist constants $c_1, c_2, c_3, c_4 \geq 0$ such that*

$$\Phi\, K = c_1\, \mathrm{M}_p^+\, K +_p c_2\, \mathrm{M}_p^-\, K +_p c_3\, \mathrm{I}_p^+\, K +_p c_4\, \mathrm{I}_p^-\, K$$

for all $K \in \mathcal{K}_o^n$.

Proof. The map

$$P \mapsto h\,(\Phi\, P, .)^p\,, \quad P \in \mathcal{P}_o^n$$

satisfies the assumptions of Theorem 4.4. Thus, we get constants $d_1, d_2, d_3, d_4 \in \mathbb{R}$ such that

$$h\,(\Phi\, P, .)^p = d_1 h\left(\mathrm{M}_p^+\, P, .\right)^p + d_2 h\left(\mathrm{M}_p^-\, P, .\right)^p + d_3 h\left(\mathrm{I}_p^+\, P, .\right)^p + d_4 h\left(\mathrm{I}_p^-\, P, .\right)^p$$

holds for all $P \in \mathcal{P}_o^n$. By continuity this is true for all $K \in \mathcal{K}_o^n$. It remains to show that $d_1, d_2, d_3, d_4 \geq 0$. To this end consider for fixed $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$ the map $s \mapsto h\,(\Phi(sK), x)^p\,,\ s > 0$. Using the homogeneity of the operators M_p^+, M_p^-, I_p^+ and I_p^- we get

$$\begin{aligned} 0 &\leq h\,(\Phi(sK), x)^p \\ &= \left(d_1 h\left(\mathrm{M}_p^+\, K, x\right)^p + d_2 h\left(\mathrm{M}_p^-\, K, x\right)^p\right) s^{n+p} + \left(d_3 h\left(\mathrm{I}_p^+\, K, x\right)^p + d_4 h\left(\mathrm{I}_p^-\, K, x\right)^p\right) s^p. \end{aligned}$$

Dividing this equation by s^p and letting $s \to 0$ we see that

$$0 \leq d_3 h\left(\mathrm{I}_p^+\, K, x\right)^p + d_4 h\left(\mathrm{I}_p^-\, K, x\right)^p.$$

Setting $K = [o, e_1]$ and $x = \pm e_1$ and using (6) shows that $d_3, d_4 \geq 0$. Similarly we see that $d_1, d_2 \geq 0$. Defining $c_i = \sqrt[p]{d_i}$ for $i = 1, \ldots, 4$ finishes the proof. $\square$

5 Main Results on $\mathcal{K}^n$

In this section our goal is the classification of all continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations on $\mathcal{K}^n$. We start with a classification of valuations from $\mathcal{P}^n$ to $C_p(\mathbb{R}^n)$ with the additional assumption that the valuation is almost simple. Here we call a valuation $\Phi\colon \mathcal{Q}^n \to A$ almost simple, if $\Phi\, K = 0$ for all $K \in \mathcal{Q}^n$ with $\dim K \leq n-2$ and for all $K \in \mathcal{Q}^n$ with $\dim K = n-1$ and $o \in \operatorname{aff} K$.

5.1 Theorem. *Let $n \geq 3$ and let $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ be an almost simple $\mathrm{SL}(n)$-covariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0, +\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$ and that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $J_y \subseteq (0, +\infty)$ such that $\{\Phi(s\widetilde{T}^{n-1})(y) : s \in J_y\}$ is not dense in $\mathbb{R}$. Then there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that*

$$\Phi\, P = c_1 h\left(\mathrm{M}_p^+\, P, .\right)^p + c_2 h\left(\mathrm{M}_p^-\, P, .\right)^p + c_3 h\left(\hat{\mathrm{M}}_p^+ P, .\right)^p + c_4 h\left(\hat{\mathrm{M}}_p^- P, .\right)^p$$

for all $P \in \mathcal{P}^n$.

Proof. By replacing T^n with $\widetilde{T}^{n-1}$ and by replacing (7) and (8) with (11) and (12), respectively, in the proof of Theorem 4.1 we see that there exist constants c_3, c_4 such that

$$\Phi(s\widetilde{T}^{n-1}) = c_3 h\left(\hat{\mathrm{M}}_p^+ (s\widetilde{T}^{n-1}), .\right)^p + c_4 h\left(\hat{\mathrm{M}}_p^- (s\widetilde{T}^{n-1}), .\right)^p$$

for $s > 0$. Note that the constants are given by

$$c_3 = \frac{\Gamma(p+1+n)}{\Gamma(p+1)}\,\Phi(\widetilde{T}^{n-1})(e_1) \quad \text{and} \quad c_4 = \frac{\Gamma(p+1+n)}{\Gamma(p+1)}\,\Phi(\widetilde{T}^{n-1})(-e_1).$$

Define $\Psi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ by

$$\Psi\, P = \Phi\, P - c_3 h\left(\hat{\mathrm{M}}_p^+ P, .\right)^p - c_4 h\left(\hat{\mathrm{M}}_p^- P, .\right)^p$$

for all $P \in \mathcal{P}^n$. Note that Ψ is an $\mathrm{SL}(n)$-covariant valuation. We use $\Psi(s\widetilde{T}^{n-1}) = 0$, the $\mathrm{SL}(n)$-covariance of Ψ, Theorem 2.1 and the assumption that Φ is almost simple to see that Ψ is simple. Now Theorem 4.1 implies that there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$\Psi\, P = c_1 h\left(\mathrm{M}_p^+\, P, .\right)^p + c_2 h\left(\mathrm{M}_p^-\, P, .\right)^p \tag{34}$$

for all $P \in \mathcal{P}_o^n$. Because M_p^+ and M_p^- are simple valuations on $\mathcal{P}^n$, Lemma 2.5 implies that (34) holds for all $P \in \mathcal{P}^n$. Using the definition of Ψ finishes the proof. □

The next lemma is the analog of Lemma 4.2 for $\mathcal{P}^n$ and will be needed in the induction step in the proof of Theorem 5.4. Again, it will rule out the existence of certain valuations.

5.2 Lemma. *Let $n \geq 2$. If $\Phi\colon \mathcal{P}_o^n \to C_p(\mathbb{R}^n)$ is an almost simple $\mathrm{GL}(n)$-covariant valuation which is continuous at the line segment $[0, e_1]$, then $\Phi = 0$.*

Proof. First note that the $\mathrm{GL}(n)$-covariance implies p-homogeneity. For $n \geq 3$ the assertion follows directly from Theorem 5.1, since $P \mapsto h\left(\mathrm{M}_p^+\, P, .\right)^p$, $P \mapsto h\left(\mathrm{M}_p^-\, P, .\right)^p$, $P \mapsto h\left(\hat{\mathrm{M}}_p^+ P, .\right)^p$ and $P \mapsto h\left(\hat{\mathrm{M}}_p^- P, .\right)^p$ are all $(n+p)$-homogeneous.

Only the case $n = 2$ remains to be proved. Similar to the steps 3 and 4 in the proof of Theorem 4.1, we see that $\Phi\,\widetilde{T}^1$ is determined by the two values $\Phi(\widetilde{T}^1)(\pm e_1)$. Now notice that

$$P \mapsto h\left(\mathrm{I}_p^+\, P, .\right)^p - h\left(\mathrm{E}_p^+\, P, .\right)^p \quad \text{and} \quad P \mapsto h\left(\mathrm{I}_p^-\, P, .\right)^p - h\left(\mathrm{E}_p^-\, P, .\right)^p,$$

are almost simple $\mathrm{GL}(n)$-covariant valuations. Using (16), the $\mathrm{GL}(n)$-covariance and Theorem 2.1 we see that by subtracting a suitable linear combination of these two operators from Φ we get a simple $\mathrm{GL}(n)$-covariant valuation Ψ. Using the same arguments as in the proof of Theorem 4.1 and using Lemma 2.5 we see that Ψ is determined by the two values $\Phi(T^2)(\pm e_1)$. Since

$$P \mapsto h\left(\mathrm{I}_p^+\, P, .\right)^p - h\left(\mathrm{E}_p^+\, P, .\right)^p + \mathrm{J}_p^+\, P - \mathrm{F}_p^+\, P$$

and

$$P \mapsto h\left(\mathrm{I}_p^-\, P, .\right)^p - h\left(\mathrm{E}_p^-\, P, .\right)^p + \mathrm{J}_p^-\, P - \mathrm{F}_p^-\, P$$

are simple $\mathrm{GL}(n)$-covariant valuations, we can use (17) to conclude that Ψ is a linear combination of these two operators. Therefore, Φ is a linear combination of the four operators above. Since the only operator in the linear hull of these operators which is continuous at $[o, e_1]$ is $\Phi = 0$ (see Lemma 3.2), the assertion follows. □

5.3 Lemma. *Let $n \geq 2$. If $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ is* $\mathrm{SL}(n)$*-covariant, then $\Phi(P)(x) = \Phi(P)(\pi_P x)$ for all $P \in \mathcal{P}^n$ and $x \in \mathbb{R}^n$, where π_P denotes the orthogonal projection onto* $\operatorname{lin} P$.

Proof. The proof is similar to the proof of Lemma 4.3. $\square$

Now we are able to classify all $\mathrm{SL}(n)$-covariant valuations from $\mathcal{P}^n$ to $C_p(\mathbb{R}^n)$ which satisfy certain continuity properties.

5.4 Theorem. *Let $n \geq 3$ and let $\Phi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ be an* $\mathrm{SL}(n)$*-covariant valuation. Assume further that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $I_y \subseteq (0, +\infty)$ such that $\{\Phi(sT^n)(y) : s \in I_y\}$ is not dense in $\mathbb{R}$ and that for every $y \in \mathbb{R}^n$ there exists a bounded open interval $J_y \subseteq (0, +\infty)$ such that $\{\Phi(s\widetilde{T}^{n-1})(y) : s \in J_y\}$ is not dense in $\mathbb{R}$. Also assume that Φ is continuous at the line segment $[0, e_1]$. Then there exist constants $c_i \in \mathbb{R}$, $i = 1 \dots, 8$ such that*

$$\begin{aligned}\Phi\, P = {} & c_1 h\left(\mathrm{M}_p^+ P, .\right)^p + c_2 h\left(\mathrm{M}_p^- P, .\right)^p + c_3 h\left(\hat{\mathrm{M}}_p^+ P, .\right)^p + c_4 h\left(\hat{\mathrm{M}}_p^- P, .\right)^p \\ & + c_5 h\left(\mathrm{I}_p^+ P, .\right)^p + c_6 h\left(\mathrm{I}_p^- P, .\right)^p + c_7\, \mathrm{J}_p^+ P + c_8\, \mathrm{J}_p^- P\end{aligned}$$

for all $P \in \mathcal{P}^n$.

Proof. Lemma 5.3 and the p-homogeneity of the functions in $C_p(\mathbb{R}^n)$ show that

$$\Phi([o, e_1])(x) = \Phi([o, e_1])(x_1 e_1) = |x_1|^p\, \Phi([o, e_1])(\operatorname{sgn}(x_1) e_1) \tag{35}$$

and

$$\Phi([e_1, 2e_1])(x) = |x_1|^p\, \Phi([e_1, 2e_1])(\operatorname{sgn}(x_1) e_1) \tag{36}$$

for all $x \in \mathbb{R}^n$. Set $c_5 = \Phi([o, e_1])(e_1)$, $c_6 = \Phi([o, e_1])(-e_1)$, $c_7 = \Phi([e_1, 2e_1])(e_1) - c_5 2^p$ and $c_8 = \Phi([e_1, 2e_1])(-e_1) - c_6 2^p$. Define $\Psi\colon \mathcal{P}^n \to C_p(\mathbb{R}^n)$ by

$$\Psi\, P = \Phi\, P - c_5 h\left(\mathrm{I}_p^+ P, .\right)^p - c_6 h\left(\mathrm{I}_p^- P, .\right)^p - c_7\, \mathrm{J}_p^+ P - c_8\, \mathrm{J}_p^- P \tag{37}$$

for all $P \in \mathcal{P}^n$. Note that Ψ is also an $\mathrm{SL}(n)$-covariant valuation. If we can show that Ψ is almost simple, then the assertion follows from Theorem 5.1.

We need to prove that $\Psi\, P = 0$ for all $P \in \mathcal{P}^n$ with $\dim P \leq n - 2$ and for all $P \in \mathcal{P}^n$ with $\dim P = n - 1$ and $o \in \operatorname{aff} P$. Using the $\mathrm{SL}(n)$-covariance we can assume without loss of generality that $P \subseteq \operatorname{lin}\{e_1, \dots, e_{n-1}\}$. We will use induction on $d = 1, \dots, n-1$ to show that $\Psi\, P = 0$ for all convex polytopes $P \subseteq \operatorname{lin}\{e_1, \dots, e_d\}$. Define $\hat{\Psi}\colon \mathcal{P}^d \to C_p(\mathbb{R}^d)$ by

$$\hat{\Psi} P = \Psi(\iota_d P) \circ \iota_d \tag{38}$$

for all $P \in \mathcal{P}^d$, where ι_d denotes the natural embedding of $\mathbb{R}^d$ in $\mathbb{R}^n$. It is easy to see that $\hat{\Psi}$ is a $\mathrm{GL}(d)$-covariant valuation. Since Ψ is $\mathrm{SL}(n)$-covariant, we have $\Psi(\{o\}) = 0$ by Lemma 5.3. For $d = 1$ the induction statement follows from (35), (36), (6), (9), (10), the $\mathrm{GL}(d)$-covariance and Lemma 5.3. For $2 \leq d \leq n-1$ the induction statement follows from the induction hypothesis, Lemma 5.2 and Lemma 5.3. This finishes the induction and completes the proof of the theorem. $\square$

Finally we can prove our desired result about continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations on $\mathcal{K}^n$. This is one direction of the second main theorem from the introduction. The other direction is trivial.

5.5 Corollary. *Let $n \geq 3$. If $\Phi\colon \mathcal{K}^n \to \mathcal{K}^n_o$ is a continuous $\mathrm{SL}(n)$-covariant L_p-Minkowski valuation, then there exist constants $c_1, c_2, c_3, c_4, c_5, c_6 \geq 0$ such that*

$$\Phi\, K = c_1 \,\mathrm{M}_p^+ K +_p c_2 \,\mathrm{M}_p^- K +_p c_3 \hat{\mathrm{M}}_p^+ K +_p c_4 \hat{\mathrm{M}}_p^- K +_p c_5 \,\mathrm{I}_p^+ K +_p c_6 \,\mathrm{I}_p^- K$$

for all $K \in \mathcal{K}^n$.

Proof. The map

$$P \mapsto h\,(\Phi\, P, .)^p\,, \quad P \in \mathcal{P}^n_o$$

satisfies the assumptions of Theorem 5.4. Thus, we get constants $d_1, \dots, d_8 \in \mathbb{R}$ such that

$$\begin{aligned} h\,(\Phi\, P, .)^p = d_1 h \left(\mathrm{M}_p^+ P, .\right)^p + d_2 h \left(\mathrm{M}_p^- P, .\right)^p + d_3 h \left(\hat{\mathrm{M}}_p^+ P, .\right)^p + d_4 h \left(\hat{\mathrm{M}}_p^- P, .\right)^p \\ + d_5 h \left(\mathrm{I}_p^+ P, .\right)^p + d_6 h \left(\mathrm{I}_p^- P, .\right)^p + d_7 \,\mathrm{J}_p^+ P + d_8 \,\mathrm{J}_p^- P \end{aligned}$$

holds for all $P \in \mathcal{P}^n$. By continuity this holds for all $K \in \mathcal{K}^n$. It remains to show that $d_1, \dots, d_6 \geq 0$ and that $d_7, d_8 = 0$. Similar to the proof of Corollary 4.5 we consider the map $s \mapsto h\,(\Phi(sK), .)\,,\ s > 0$ for fixed $K \in \mathcal{K}^n$ to see that both

$$\left(d_1 h \left(\mathrm{M}_p^+ K, .\right)^p + d_2 h \left(\mathrm{M}_p^- K, .\right)^p + d_3 h \left(\hat{\mathrm{M}}_p^+ K, .\right)^p + d_4 h \left(\hat{\mathrm{M}}_p^- K, .\right)^p\right)^{\frac{1}{p}}$$

and

$$\left(d_5 h \left(\mathrm{I}_p^+ K, .\right)^p + d_6 h \left(\mathrm{I}_p^- K, .\right)^p + d_7 \,\mathrm{J}_p^+ K + d_8 \,\mathrm{J}_p^- K\right)^{\frac{1}{p}} \tag{39}$$

are nonnegative and sublinear. Using (7), (8), (11), (12) and (6) we get that $d_1, \dots, d_6 \geq 0$. Now set $K = \mathrm{conv}\{e_1, e_2, e_1 + e_2\}$ in (39). Inserting the directions e_1, e_2 and $e_1 + e_2$ and using the subadditivity and (13) gives us

$$(2^p d_5 + d_7)^{\frac{1}{p}} \leq (d_5)^{\frac{1}{p}} + (d_5)^{\frac{1}{p}} = 2\,(d_5)^{\frac{1}{p}}\,.$$

Therefore we get $d_7 \leq 0$. Meanwhile, setting $K = \widetilde{T}^1$ in (39) and inserting the directions e_1, $e_1 + e_2$ and $2e_1 + e_2$ gives us by (14)

$$0 \leq (d_5)^{\frac{1}{p}} + (d_5 + d_7)^{\frac{1}{p}} - (2^p d_5 + d_7)^{\frac{1}{p}}\,.$$

The right hand side of this equation is equal to 0 for $d_7 = 0$. By differentiating the right hand side with respect to d_7 and by using $p > 1$ we see that it is strictly monotone. Therefore we get $d_7 \geq 0$. The last two arguments show that $d_7 = 0$. Similary we see that $d_8 = 0$. Setting $c_i = \sqrt[p]{d_i}$ for $i = 1, \dots, 6$ finishes the proof. $\square$

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